Boundary methods for Dirichlet problems of Laplace’s equation in elliptic domains with elliptic holes

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A B S T R A C T
Recently, the null field method (NFM) is proposed by J.T. Chen with his groups. In NFM, the fundamental solutions (FS) with the field nodes Q outside of the solution domains are used in the Green formulas. In this paper, the NFM is developed for the elliptic domains with elliptic holes. First, the FS is expanded by the infinite series in elliptic coordinates. When the Fourier approximations of the boundary conditions on the elliptic boundaries are chosen, the explicit algebraic equations are derived, and the semi-analytic solutions can be found. Next, the interior field method (IFM) is developed, which is equivalent to the NFM when the field nodes approach the domain boundary. Moreover, the collocation Trefftz method (CTM) is also employed by using the particular solutions in elliptic coordinates. The CTM is the simplest algorithm, has no risk of degenerate scales, and can be applied to non-elliptic domains. Numerical experiments are carried out for elliptic domains with one elliptic hole by the IFM, the NFM and the CTM. In summary, for Laplace’s equation in elliptic domains, a comparative study of algorithms, errors, stability and numerical results is explored in this paper for three boundary methods: the NFM, the IFM and the CTM.

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1. Introduction

The particular solutions (PS) and fundamental solutions (FS) in polar coordinates can be found in many textbooks, but with much less coverage in elliptic coordinates [12, 28, 29]. Since the elliptic domains with elliptic holes may be found in some engineering problems, the PS and the FS expansions in elliptic coordinates are essential for numerical computations. In this paper, for Laplace’s equation in elliptic coordinates, the null field method (NFM), the interior field method (IFM) and the collocation Trefftz method (CTM) are developed with explicit algorithms.

Let us briefly mention some related references. For circular domains with circular holes, there exist many papers by boundary methods. In Barone and Caulk [2, 3] and Caulk [6] the Fourier series are used for the circular domains for boundary integral equations, and in Bird and Steele [4] the simple algorithms are used as the collocation Trefftz method as in [27]. In Ang and Kang [1], complex boundary elements are studied. For circular and elliptic domains, the NFM is developed by Chen’s group, see [8–13]. Recently, the explicit collocation equations and semi-analytic solutions are derived in Li et al. [25], and the conservative schemes are developed in Lee et al. [19]. The new interior field method (IFM) is proposed in [17], which is equivalent to the specific NFM at the field nodes on the domain boundary. The algorithm singularity resulting from degenerate scales is analyzed in [21] for the NFM in circular domains. Moreover, the NFM and the IFM for Neumann problems are developed in [20] for Laplace’s equation in circular domains. For circular and elliptic domains, the study of J.T. Chen and YZ. Chen in [8–16] seems to be more focused on the degenerate scale problems. Our efforts are mainly paid for establishing effective algorithms by following [23, 27]. Then, this paper
is devoted to effective boundary methods for Dirichlet problems of Laplace’s equation in elliptic domains.

This paper is organized as follows. In the next section, the elliptic coordinates are introduced, and the expansions of the FS in elliptic coordinates are derived in detail. In Section 3, the null field method (NFM) and the collocation Trefftz method (CTM) are developed, respectively. In Section 6, numerical experiments are carried out, and in the last section, a few concluding remarks are addressed.

2. Elliptic coordinates

2.1. Definitions

First, we introduce the elliptic coordinates for the boundary methods to be described later. The elliptic coordinates are defined in [28,29] by

\[ x = \sigma \cosh \rho \cos \theta, \quad y = \sigma \sinh \rho \sin \theta, \quad (2.1) \]

where \( \sigma > 0 \), and two coordinates \((\rho, \theta)\) have the ranges: \( 0 \leq \rho < \infty \) and \( 0 \leq \theta \leq 2\pi \). When \( \rho = \rho_0 = 0, \) Eq. (2.1) leads to an ellipse

\[ \frac{x^2}{\sigma^2 \cosh^2 \rho_0} + \frac{y^2}{\sigma^2 \sinh^2 \rho_0} = 1, \quad (2.2) \]

with two semiaxes (see Fig. 1)

\[ a = \sigma \cosh \rho_0, \quad b = \sigma \sinh \rho_0. \quad (2.3) \]

When the two semiaxes are known, we have

\[ \rho_0 = \tanh^{-1} \frac{b}{a} \quad (2.4) \]

where \( \tanh^{-1} \eta = \xi \) is the inverse hyperbolic function, defined by \( \eta = \tanh \xi \). Since

\[ \sqrt{a^2 - b^2} = \sqrt{\sigma^2 \cosh^2 \rho_0 - \sigma^2 \sinh^2 \rho_0} = \sigma, \quad (2.5) \]

two foci of the ellipse are at \((\pm \sigma, 0)\).

Define the eccentricity by

\[ e^* = \frac{\sqrt{a^2 - b^2}}{a} \quad (2.6) \]

For a given \( \sigma \) in (2.1), when \( \rho_0 \to 0 \), we have \( a \to \sigma \) from (2.3) and \( e^* \to 1 \) from (2.6), to indicate the full eccentricity. When the foci \((\pm \sigma, 0)\) approach \((\pm a, 0)\), the ellipse degenerates to a strip on the \( x \)-axis with \(-\sigma \leq x \leq \sigma \). Unfortunately, the null field method (NFM) fails in the strip case, because the boundary derivatives expressed by Fourier expansions are valid only for smooth solutions, but not for singular solutions. In contrast, the collocation Trefftz method (CTM) in Section 5 is well suited to singularity problems, since the singular solutions can be chosen wherein. When \( \rho_0 \) is large, there occurs nearly the zero eccentricity \( e^* \approx 0 \) from (2.6), and then the ellipse is approximated to a circle.

On the other hand, when \( \theta = \theta_0 \), Eq. (2.1) leads to a hyperbola (also see Fig. 1)

\[ \frac{x^2}{\sigma^2 \cos^2 \theta_0} - \frac{y^2}{\sigma^2 \sin^2 \theta_0} = 1, \quad (\theta_0 \neq \frac{k\pi}{2}, \; k = 0, 1, 2, 3, 4, \quad (2.7) \]

where \( a = \sigma \cos \theta_0 \) and \( b = \sigma \sin \theta_0 \) are two semiaxes, and two foci are at \((\pm a^*, 0)\) with \( a^* = \sqrt{a^2 + b^2} \). The coordinate curves of (2.1) are elliptic and hyperbolic in (2.2) and (2.7), respectively, which are orthogonal to each other.

2.2. Coordinate transformations

For explicit algebraic equations, we need the coordinate transformations between two different elliptic coordinates. First, we seek the transformation between the Cartesian coordinates \((x,y)\) and the elliptic coordinates \((\rho, \theta)\). We obtain \((x,y)\) from \((\rho, \theta)\) directly from (2.1). Below, we seek the inverse transformation, i.e., from \((x,y)\) to \((\rho, \theta)\).

From (2.1) we have

\[ \frac{x^2}{\sigma^2 \cosh^2 \rho} + \frac{y^2}{\sigma^2 \sinh^2 \rho} = 1. \quad (2.8) \]

Let \( v = \sinh \rho \). Since \( \cosh^2 \rho = 1 + \sinh^2 \rho = 1 + v \), we have from (2.8)

\[ \frac{x^2}{1 + v} + \frac{y^2}{v} = \sigma^2. \quad (2.9) \]

This gives

\[ \sigma^2 v^2 - v(x^2 + y^2 - \sigma^2) - y^2 = 0. \quad (2.10) \]

Two roots of the above equation are found as

\[ v^\pm = \frac{1}{2\sigma^2}\left\{ (x^2 + y^2 - \sigma^2) \pm \sqrt{(x^2 + y^2 - \sigma^2)^2 + 4\sigma^2 y^2} \right\}. \quad (2.11) \]

Since only the positive solution \( v^+ \) is valid, we have from (2.11)

\[ \sinh^2 \rho = v^+ = \frac{1}{2\sigma^2}\left\{ (x^2 + y^2 - \sigma^2) + \sqrt{(x^2 + y^2 - \sigma^2)^2 + 4\sigma^2 y^2} \right\}. \quad (2.12) \]

Define the function

\[ F(x,y; \sigma) = \frac{1}{\sqrt{2\sigma}} \sqrt{(x^2 + y^2 - \sigma^2) + \sqrt{(x^2 + y^2 - \sigma^2)^2 + 4\sigma^2 y^2}}. \quad (2.13) \]

Hence, we obtain the explicit transformation formulas from \((x,y)\) to \((\rho, \theta)\)

\[ \rho = \sinh^{-1}(F(x,y; \sigma)), \quad \theta = \cos^{-1}\left(\frac{x}{\sigma \cosh \rho}\right). \quad (2.14) \]

Below, consider the interior ellipse at center \((x_1, y_1)\), and the directions of two semiaxes are rotated by a counter-clockwise angle \( \Theta \), see Fig. 2. When the other Cartesian coordinates \(X\Theta Y\) are located from the standard Cartesian coordinates by rotating a counter-clockwise angle \( \Theta \in [0, \pi] \), there exist the relations of the Cartesian coordinates

\[ \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \Theta & \sin \Theta \\ -\sin \Theta & \cos \Theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \quad (2.15) \]
The global elliptic system with coordinates \((\rho, \theta)\) is located at center \(O = (0, 0)\). Assume that the local elliptic system with coordinates \((\overline{\rho}, \overline{\theta})\) and \(\sigma_1\) is located at \((x_1, y_1)\), and the directions of two semiaxes are the same as those of \(S_R\) in Fig. 2. We derive the coordinate transformations for the same node from \((\rho, \theta)\) to \((\overline{\rho}, \overline{\theta})\), and vice versa.

First, when the global elliptic coordinates \((\rho, \theta)\) are known for node \(P\), the Cartesian coordinates in \(XOY\) are given by

\[
\begin{align*}
(x, y) &= (\cos \theta \sin \sigma_1, - \sin \theta \cos \sigma_1), \\
\rho &= \sinh^{-1}(F(x, y; \sigma_1)), \\
\overline{\rho} &= \cos^{-1}\left(\frac{x}{\sigma_1 \cosh \rho}\right),
\end{align*}
\]

where \(F(x, y; \sigma)\) are defined in (2.13). Then, we find the transformation from \((\rho, \theta)\) to \((\overline{\rho}, \overline{\theta})\), denoted as

\[
T : (\rho, \theta) \rightarrow (\overline{\rho}, \overline{\theta}),
\]

where \((\overline{\rho}, \overline{\theta})\) are given by (2.19), and \((x', y')\) by (2.17) and (2.18) yielding

\[
\begin{align*}
\begin{pmatrix} x' \\ y' \end{pmatrix} &= \left(\begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array}\right) \begin{pmatrix} x-x_1 \\ y-y_1 \end{pmatrix},
\end{align*}
\]

Next, when the local elliptic coordinates \((\overline{\rho}, \overline{\theta})\) for node \(P\) are known, the Cartesian coordinates \((x', y')\) in \(XOY\) are given by

\[
\begin{align*}
\begin{pmatrix} x' \\ y' \end{pmatrix} &= \left(\begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array}\right) \begin{pmatrix} \sigma_1 \cosh \rho \cos \theta - x_1 \\ \sigma_1 \sinh \rho \sin \theta - y_1 \end{pmatrix},
\end{align*}
\]

where \(F(x', y'; \sigma)\) are defined in (2.13).

2.3. Derivatives and integrals under elliptic coordinates

Let us consider the integrals along the elliptic boundary \(\ell_P\) defined by (2.1) with \(\rho = \text{constant}\). We have

\[
ds = \sqrt{\frac{\partial x}{\partial \rho}^2 + \frac{\partial y}{\partial \rho}^2} \, d\theta = \sigma_1 \sinh \rho \, d\theta.
\]

For \(\ell_P\) in Fig. 1, denote \(ds = \sigma_1 \sinh \rho \, d\theta\), where

\[
\sigma_1 = \sqrt{\cosh^2 \rho + \sin^2 \rho}.
\]

The integral along \(\ell_P\) is given by

\[
\int_{\ell_P} \Phi(\rho, \theta) \, ds = \int_0^{2\pi} \Phi(\rho, \theta) \sigma_1 \sinh \rho \, d\theta.
\]

Next, consider the normal derivatives \(\frac{\partial}{\partial \nu}\) of \(\ell_P\). The normal direction to \(\ell_P\) is just along the hyperbola \(\theta = \text{constant}\), since the elliptic coordinates are orthogonal to each other. The differentiation along the hyperbola is given by

\[
\frac{\partial}{\partial \nu} = \frac{\partial}{\partial \rho} \left(\sigma_1 \cosh \rho \, d\theta\right) = \sigma_1 \rho \, d\theta.
\]

Then the normal derivatives to \(\ell_P\) are obtained by

\[
\frac{\partial \Phi}{\partial \nu} = \frac{\partial \Phi}{\partial \rho} \frac{1}{\sigma_1 \rho} = \frac{\partial \Phi}{\partial \rho} \frac{1}{\sigma_1 \rho} \frac{1}{\sigma_1 \cosh \rho}.
\]

Suppose that the highly smooth solutions of Laplace’s equations can be expressed by a convergent series in terms of elliptic coordinates, as

\[
u(\rho, \theta) = \alpha_0(\rho) + \sum_{n=1}^{\infty} \left(\alpha_n(\rho) \cos n\theta + \beta_n(\rho) \sin n\theta\right)
\]

where \(\alpha_n(\rho) = a_n\) and \(\beta_n(\rho) = b_n\) on the elliptic boundary \(\ell_P\). Hence the normal derivatives of \(\ell_P\) can be denoted from (2.32) and (2.33)\(^2\)

\[
\frac{\partial u(\rho, \theta)}{\partial \nu} = -\frac{1}{\sigma_1 \rho} \left(\alpha_0(\rho) + \sum_{n=1}^{\infty} \left(\alpha_n(\rho) \cos n\theta + \beta_n(\rho) \sin n\theta\right)\right)
\]

\(^2\) Note that Eq. (2.34) is invalid for singularity solutions.
where $\alpha_n^*(\rho) = p_n$ and $\beta_n^*(\rho) = q_n$ with $\rho = \text{constant}$.

### 2.4. Infinite series expansions of the fundamental solutions

The series expansions of the fundamental solutions (FS), \( \ln|z-z_0| \), are essential to the NFM. In this subsection, we will derive the FS expansions in elliptic coordinates. The elliptic coordinates (2.1) can be expressed by the complex

\[ z = x + iy = \sigma \cos w, \]

where \( w = \rho + i\theta \) and \( i = \sqrt{-1} \). Denote the other node in the same elliptic coordinate system (2.35) by

\[ z_0 = x_0 + iy_0 = \sigma \cos w_0, \]

where \( w_0 = r + i\phi \). Denote \( x = (\rho, \theta) \) and \( y = (r, \phi) \). We have a lemma.

**Lemma 2.1.** In the same elliptic coordinate system, let \( z \) and \( z_0 \) denote two nodes given in (2.35) and (2.36), respectively. There exist the series expansions for the FS

\[ \ln|z-z_0| = U'(x, y), \]

\[ = r + \ln \left( \frac{\rho}{w} \right) - 2 \sum_{n=1}^{\infty} \frac{1}{n} \left( \cos n\rho \cos n\theta \cos n\phi + \sinh n\rho \sin n\theta \sin n\phi \right), \quad \rho < r, \]

\[ \ln|z-z_0| = U''(x, y), \]

\[ = r - \ln \left( \frac{\rho}{w} \right) - 2 \sum_{n=1}^{\infty} \frac{1}{n} \left( \cos n\rho \cos n\theta \cos n\phi + \sinh n\rho \sin n\theta \sin n\phi \right), \quad \rho > r, \]

where the superscripts “e” and “r” designate the exterior and the interior field nodes, respectively.

**Proof.** Although the basic ideas of the proof follow from Morse and Feshbach [28], the arguments below are more comprehensive and beneficial to real application. We have from (2.35) and (2.36)

\[ \ln|z-z_0| = \frac{1}{2} \ln|z-z_0|^2 = \frac{1}{2} \ln|\sigma|^2 \cos w - \cos w_0|^2 \]

\[ = \frac{1}{2} \ln \left\{ \left( \frac{\rho}{w} \right) \right\} \left[ e^w + e^{-w} - e^{w_0} - e^{-w_0} \right] \]

\[ = \ln^2 \left\{ \frac{\rho}{w} \right\} + \ln \left\{ |e^w + e^{-w} - e^{w_0} - e^{-w_0}| \right\}. \]

There exists the equality

\[ e^w + e^{-w} - e^{w_0} - e^{-w_0} = 4 \sinh \frac{w+w_0}{2} \sinh \frac{w-w_0}{2} \]

Then we have from (2.39)

\[ \ln|z-z_0| = \ln^2 \left\{ \frac{\rho}{w} \right\} + \ln \left\{ 2 \sinh \frac{w+w_0}{2} \right\} + \ln \left\{ 2 \sinh \frac{w-w_0}{2} \right\}. \]

For \( z = re^{i\theta} \), we have \( \ln z = \ln r + i\theta \), and then

\[ \ln|z| = \ln r = \Im \ln z, \]

where \( \Im \ln z \) is real part of complex \( z \). From (2.42) we have

\[ \ln \left\{ 2 \sinh \frac{w+w_0}{2} \right\} = \ln \left\{ e^{w+w_0}/2(1 - e^{-(w+w_0)}) \right\} \]

\[ = \Im \left\{ \frac{w+w_0}{2} \right\} + \Im \left\{ \ln(1 - e^{-(w+w_0)}) \right\} \]

\[ = \Im \left\{ \frac{w+w_0}{2} \right\} - \Im \left\{ \sum_{n=1}^{\infty} \frac{1}{n} e^{-n(w+w_0)} \right\} \]

where we have used the series

\[ \ln(1-z) = - \sum_{n=1}^{\infty} \frac{z^n}{n}, \quad |z| < 1. \]

For the last term on the right hand side in (2.41), when \( \rho < r \), we have similarly

\[ \ln \left\{ \frac{2 \sinh \frac{w_0-w}{2}}{2} \right\} = \Im \left\{ \frac{w+w_0}{2} \right\} + \Im \left\{ \ln(1 - e^{-(w+w_0)}) \right\} \]

\[ = \Im \left\{ \frac{w+w_0}{2} \right\} - \Im \left\{ \sum_{n=1}^{\infty} \frac{1}{n} e^{-n(w+w_0)} \right\} \]

Combining (2.43) and (2.45) gives

\[ \ln \left\{ \frac{2 \sinh \frac{w+w_0}{2}}{2} \right\} + \Im \left\{ \frac{2 \sinh \frac{w-w_0}{2}}{2} \right\} \]

\[ = \Re \left\{ w_0 \right\} - \Re \left\{ \sum_{n=1}^{\infty} \frac{1}{n} e^{-n(w+w_0)} \right\} \]

\[ = r - \sum_{n=1}^{\infty} \frac{1}{n} \left[ e^{-n(w+r+\rho+\theta+\phi)} + e^{-n(r+c\theta+v \phi)} \right] \]

\[ = r - \sum_{n=1}^{\infty} \frac{1}{n} \left[ \cos n\rho \cos n\theta \cos n\phi + \sinh n\rho \sin n\theta \sin n\phi \right] \]

\[ = 4 \sinh \frac{w+w_0}{2} \sinh \frac{w-w_0}{2} \]

The first series expansion (2.37) follows from (2.41) and (2.46). On the other hand, when \( \rho > r \), we replace (2.45) by

\[ \ln \left\{ \frac{2 \sinh \frac{w_0-w}{2}}{2} \right\} = \Re \left\{ \frac{w_0-w}{2} \right\} - \Re \left\{ \sum_{n=1}^{\infty} \frac{1}{n} e^{-n(w+w_0)} \right\}, \]

and the proof for the second series expansion (2.38) is similar. This completes the proof of Lemma 2.1.

For the elliptic boundary \( \epsilon \), with \( r = \text{const} \), we have the normal derivatives based on Lemma 2.1 and (2.32)

\[ \frac{\partial U'(x, y)}{\partial \nu} = \frac{1}{\sigma \tau(\phi)} \frac{\partial U'(x, y)}{\partial \tau}, \]

\[ \frac{\partial U'(x, y)}{\partial \nu} = \frac{1}{\sigma \tau(\phi)} \frac{\partial U'(x, y)}{\partial \tau}, \]

\[ \rho < r, \]

\[ \frac{\partial U'(x, y)}{\partial \nu} = \frac{1}{\sigma \tau(\phi)} \frac{\partial U'(x, y)}{\partial \tau}, \]

\[ \rho > r, \]

where \( \tau(\phi) = \sqrt{\sinh^2 r + \sin^2 \theta} \).
3. The null field method

3.1. Basic algorithms

Denote the large ellipse $S_R$ with $\rho = R$, where the elliptic coordinates $(\rho, \theta)$ are given by

$$x = \sigma_1 \rho \cos \theta, y = \sigma_2 \rho \sin \theta,$$  \hspace{1cm} (3.1)

with the origin $(0,0)$. Also denote a small ellipse $S_{R_k} \subset S_R$ with $\mathcal{F} = R_1$, where the other elliptic coordinates $(\tau, \nu)$ are denoted by

$$x = \sigma_1 \rho \cos \mathcal{F} + x_1, \quad y = \sigma_1 \rho \sin \mathcal{F} + y_1.$$  \hspace{1cm} (3.2)

This Cartesian system $(\mathcal{F}, \nu)$ with the origin $(x_1, y_1)$ is rotated from the $x$-axis, by a counter-clockwise angle $\mathcal{F}$ as in Fig. 2. The coordinate transformations between $(\rho, \theta)$ and $(\tau, \nu)$ are given in (2.20) and (2.25). Denote the solution domain by $S = S_R \setminus S_K$ and its boundary as $\mathcal{S} = S_R \cup S_K$. In this paper, we only discuss the Dirichlet problem

$$\Delta u = \frac{\partial^2 u}{\partial \tau^2} + \frac{\partial^2 u}{\partial \nu^2} = 0 \text{ in } S,$$  \hspace{1cm} (3.3)

$$u = f \text{ on } \mathcal{S}_R, u = g \text{ on } \mathcal{S}_K,$$  \hspace{1cm} (3.4)

where $f \in H^2(\mathcal{S}_R)$ and $g \in H^2(\mathcal{S}_K)$. Assume that the solution is highly smooth in $S$. Based on (2.33) and (2.34), there exist the approximations of Fourier expansions on the exterior elliptic boundary $\mathcal{S}_R$

$$u = u_0 + \sum_{k=1}^{M} \{a_k \cos k\theta + b_k \sin k\theta\} \text{ on } \mathcal{S}_R,$$  \hspace{1cm} (3.5)

$$\frac{\partial u}{\partial \tau} = q_0 = \frac{1}{\sigma_1 \tau_0(\nu)} \left\{p_0 + \sum_{k=1}^{M} \{p_k \cos k\theta + q_k \sin k\theta\} \right\} \text{ on } \mathcal{S}_R,$$  \hspace{1cm} (3.6)

where $a_k, b_k, p_k$ and $q_k$ are coefficients, and $\tau_0(\nu) = \sqrt{\sinh^2 \nu + \sin^2 \theta}$. Also assume the approximations on the interior elliptic boundary $\mathcal{S}_R$

$$\mathcal{U} = \mathcal{U}_0 = \mathcal{U}_0 + \sum_{k=1}^{N} \{\mathcal{U}_k \cos k\theta + \mathcal{U}_k \sin k\theta\} \text{ on } \mathcal{S}_R,$$  \hspace{1cm} (3.7)

$$\frac{\partial \mathcal{U}}{\partial \nu} = -\frac{\partial \mathcal{U}}{\partial \nu} = -\frac{1}{\sigma_2 \tau_0(\nu)} \left\{p_0 + \sum_{k=1}^{N} \{p_k \cos k\theta + q_k \sin k\theta\} \right\} \text{ on } \mathcal{S}_R,$$  \hspace{1cm} (3.8)

where $\mathcal{U}$ is the exterior normal to $\mathcal{S}_R$, $\mathcal{U}_k, \mathcal{V}_k, \mathcal{P}_k$ and $\mathcal{Q}_k$ are coefficients, and $\tau_0(\nu) = \sqrt{\sinh^2 \nu + \sin^2 \nu}$. Denote two nodes, $x = Q = (x, y) = (\rho, \theta)$ and $y = P = (\xi, \eta) = (\rho, \phi)$, where $(\rho, \theta)$ and $(\rho, \phi)$ are the corresponding elliptic coordinates. In the Green representation formula, there exist three different equations for different field (i.e., observation) nodes $Q = Q(x)$ (see[5])

$$\frac{1}{2\pi^2} \int_{\mathcal{S}_R \cup \mathcal{S}_K} \left( \ln |PQ| \frac{\partial u(y)}{\partial \nu} - u(y) \frac{\partial |PQ|}{\partial \nu} \right) d\sigma_y = \left\{ \begin{array}{ll} -u(Q), & Q \in S, \\ -\frac{1}{2u}(Q), & Q \in \mathcal{S}, \\ 0, & \text{otherwise}, \end{array} \right.$$  \hspace{1cm} (3.9)

where $P(y) \in \mathcal{S}$. When $\ln |PQ|$ is replaced by its series expansions $U(x, y)$, the third equation of (3.9) leads to

$$\int_{\mathcal{S}_R \cup \mathcal{S}_K} U(x, y) \frac{\partial u(y)}{\partial \nu} d\sigma_y = \int_{\mathcal{S}_R \cup \mathcal{S}_K} u(y) \frac{\partial U(x, y)}{\partial \nu} d\sigma_y, \quad x \in S',$$  \hspace{1cm} (3.10)

where $S'$ is the complementary domain of $S \cup \mathcal{S}_R \cup \mathcal{S}_K$.

3.2. Explicit algebraic equations

By following [25], let us derive the explicit algebraic equations. First, consider the exterior field nodes $x = (\rho, \theta)$ with $\rho > R = R$. For $(R, \phi)$ on $\mathcal{S}_R$, by substituting (3.5), (3.6), (2.38), and (2.49) into (3.10), we have the following integrals from orthogonality of Fourier series:

$$\int_{\mathcal{S}_R} u(y) \frac{\partial U(x, y)}{\partial \nu} d\sigma_y = \int_0^{2\pi} \left\{ a_0 + \sum_{k=1}^M \{a_k \cos k\phi + b_k \sin k\phi\} \right\} \times \left\{ \begin{array}{ll} -2 \sigma_0 \tau_0(\phi) \sum_{n=1}^\infty e^{-nk\phi} \sinh nR \cos n\phi \cos n\phi \\ + \cos nR \cos n\phi \sin n\phi \end{array} \right\} d\phi,$$  \hspace{1cm} (3.11)

and

$$\int_{\mathcal{S}_R} U(x, y) \frac{\partial u(y)}{\partial \nu} d\sigma_y = \int_0^{2\pi} \frac{1}{\sigma_0 \tau_0(\phi)} \left\{ p_0 + \sum_{k=1}^M \{p_k \cos k\phi + q_k \sin k\phi\} \right\} \times \left\{ \begin{array}{ll} \rho + \ln \left(\frac{\sigma_0}{2}\right) - 2 \sum_{n=1}^\infty \frac{1}{n} \cos n\phi \\ \rho + \ln \left(\frac{\sigma_0}{2}\right) - 2 \sum_{n=1}^\infty \frac{1}{n} \sinh nR \cos n\phi \cos n\phi \\ + \cos nR \cos n\phi \sin n\phi \end{array} \right\} d\phi,$$  \hspace{1cm} (3.12)

where $\tau_0(\phi) = \sqrt{\sin^2 \phi + \sin^2 \phi}$. Note that in (3.11) and (3.12), the factors, as $\sigma_0 \tau_0(\phi)$, in the integrand are canceled with each other. Since the exterior field node $(\rho, \theta)$ with $\rho > R$ to $\mathcal{S}_R$ is also an exterior field node to $\mathcal{S}_K$, we may obtain from (3.11) and (3.12) for $\mathcal{F} > R_1$, similarly

$$\int_{\mathcal{S}_K} u(y) \frac{\partial U(x, y)}{\partial \nu} d\sigma_y = \int_{\mathcal{S}_K} u(y) \frac{\partial U(x, y)}{\partial \nu} d\sigma_y = 2\pi \sum_{k=1}^N \sum_{k=1}^M e^{-k\rho} \left\{ \mathcal{U}_k \sinh kR \cos k\theta + \mathcal{U}_k \cosh kR \sin k\theta \right\},$$  \hspace{1cm} (3.13)

$$\int_{\mathcal{S}_K} U(x, y) \frac{\partial u(y)}{\partial \nu} d\sigma_y = \int_{\mathcal{S}_K} U(x, y) \frac{\partial u(y)}{\partial \nu} d\sigma_y = 2\pi \left[ \rho + \ln \left(\frac{\sigma_0}{2}\right) \right] P_0 + 2\pi \sum_{k=1}^M \sum_{k=1}^N e^{-k\rho} \left\{ \mathcal{U}_k \sinh kR \cos k\theta + \mathcal{U}_k \cosh kR \sin k\theta \right\},$$  \hspace{1cm} (3.14)

where we have used $\frac{\partial}{\partial \nu} = -\frac{\partial}{\partial \phi}$. Combining (3.10) and (3.11)–(3.14) gives the first NFM equation for exterior nodes

$$\mathcal{L}_{ext}(\rho, \theta; \mathcal{F}, \nu) = -\sum_{k=1}^M e^{-k\rho} \left\{ \mathcal{U}_k \sinh kR \cos k\theta + \mathcal{U}_k \cosh kR \sin k\theta \right\} + \sum_{k=1}^N e^{-k\rho} \left\{ \mathcal{U}_k \sinh kR \cos k\theta + \mathcal{U}_k \cosh kR \sin k\theta \right\},$$  \hspace{1cm} (3.15)
\[ + q_0 \sinh kR \sin k\theta \right \} + \left [ \rho + \ln \left ( \frac{\sigma_1}{2} \right ) \left ] \frac{\rho_0}{\rho} - \sum_{k=1}^{N} \frac{1}{k} e^{-\sqrt{k}} \left \{ \pi_k \cosh kr_1 \cos k\theta + \pi_1 \sin k\theta \right \} \right \} = 0, \quad \rho > R, \tag{3.15} \]

where the common factor \(2\pi\) has been canceled from both sides.

Second, consider the other exterior field nodes \(x = (\sigma, \theta)\) with \(\sigma < \tau = R_1\). For the small ellipse \(\sigma R_1\), we have from (3.7), (3.8), (2.37) and (2.48)

\[
\left \{ \begin{array}{l}
\int_{\sigma_1} u(x, y) \frac{\partial U(x, y)}{\partial x} dy = - \int_{\sigma_1} u(y) \frac{\partial U(x, y)}{\partial x} dy \\
= - \int_{\sigma} \left \{ \frac{1}{\sigma_1 \tau_1(t)} \left \{ \pi_0 + \sum_{k=1}^{N} \left [ \left \{ \cos \eta \tau \cos \eta \rho \sin \eta \rho \right \} \sigma_{\rho 1}(\eta(t)) \right \} \right \} e^{-\sqrt{k}} \\
\times \left \{ \left [ R_1 + \ln \left ( \frac{\sigma_1}{2} \right ) \right ] \frac{\rho_0}{\rho} - \sum_{k=1}^{N} \frac{1}{k} e^{-\sqrt{k}} \left \{ \pi_k \cosh kr_1 \cos k\theta + \pi_1 \sin k\theta \right \} \right \} \right \} = 0. \quad \rho < R_1. \tag{3.20} \end{array} \right .
\]

Note that in (3.15) and (3.20), the transformations between \((\rho, \theta)\) and \((\sigma, \theta)\) can be found from (2.20) and (2.25). Hence, Eqs. (3.15) and (3.20) are called the explicit algebraic equations of the NFM in elliptic coordinates in this paper.

### 3.3. Semi-analytic solutions

Since the Dirichlet conditions are given on \(\sigma S_k\) and \(\sigma S_k\), only \(2(M+N)+2\) coefficients in (3.15) and (3.20) are unknown. We may choose \(2(M+N)+2\) field nodes. From [21], it is better to choose the uniform nodes on the exterior and the interior ellipses

\[(\rho, \theta) = (R + \epsilon, j \Delta \theta), \quad j = 0, 1, \ldots, 2M, \tag{3.21} \]

\[(\sigma, \theta) = (R_1 - \epsilon, j \Delta \sigma), \quad j = 0, 1, \ldots, 2N. \tag{3.22} \]

where \(\epsilon \geq 0, \quad 0 \leq \epsilon < R_1, \quad \Delta \theta = \frac{\pi}{2N} \) and \(\Delta \sigma = \frac{\pi}{2N} \). Denote the explicit equations (3.15) and (3.20) by

\[ L_{\text{int}}(\rho, \theta; \sigma, \theta) = 0, \quad L_{\text{ext}}(\rho, \theta; \sigma, \theta) = 0. \tag{3.23} \]

We obtain \(2(M+N)+2\) collocation equations

\[ L_{\text{int}}(R + \epsilon, j \Delta \theta; \rho, \theta) = 0, \quad j = 0, 1, \ldots, 2M, \tag{3.24} \]

\[ L_{\text{ext}}(R_1 - \epsilon, j \Delta \sigma; \rho, \theta) = 0, \quad j = 0, 1, \ldots, 2N, \tag{3.25} \]

where the corresponding coordinates \((\rho, \theta)\) and \((\sigma, \theta)\) can be evaluated from \((R + \epsilon, j \Delta \theta)\) and \((R_1 - \epsilon, j \Delta \sigma)\), based on the coordinate transformations in Section 2.2. Eqs. (3.24) and (3.25) lead to the following linear equations:

\[ \mathbf{A}x = b. \tag{3.26} \]

where the matrices \(\mathbf{A} \in \mathbb{R}^{n \times n}\), the vector \(\mathbf{x} \in \mathbb{R}^n\) = \([x_0, x_1, \ldots, x_n]\) and \(n = 2(M+N)+2\). The unknown coefficients can be obtained by solving (3.26) if \(\mathbf{A}\) is nonsingular.

Once all the coefficients are known, based on the first equation of the Green formula (3.9), the solution at the interior nodes, \(x = (\rho, \theta) \in S\), is expressed by

\[ u(x) = u(\rho, \theta) = - \frac{1}{2\pi} \int_{\sigma} \left \{ u(x, y) \frac{\partial U(x, y)}{\partial \rho} - u(y) \frac{\partial U(x, y)}{\partial \rho} \right \} d\sigma y + \int_{\sigma} \left \{ u(x, y) \frac{\partial U(x, y)}{\partial \theta} - u(y) \frac{\partial U(x, y)}{\partial \theta} \right \} d\sigma y, \quad x \in S. \tag{3.27} \]

We can then obtain the solution for \((\rho, \theta) \in S\) similarly

\[ u(\rho, \theta) = - \frac{1}{2\pi} \int_{0}^{2\pi} \left \{ p_0 + \sum_{k=1}^{M} [p_k \cos k\rho + q_k \sin k\rho] \right \} \left \{ R + \ln \left ( \frac{\sigma_1}{2} \right ) - 2 \sum_{n=1}^{N} \frac{1}{n} e^{-\sqrt{n}} \left \{ \cos \eta \tau \cos \eta \rho \sin \eta \rho \right \} \right \} d\sigma \]

Combining (3.10) and (3.16)–(3.19) gives the second NFM equation for exterior nodes

\[ L_{\text{int}}(\rho, \theta; \sigma, \theta) = - \pi_0 - \sum_{k=1}^{N} e^{-\sqrt{k}} \left \{ \pi_k \cosh kr_1 \cos k\theta + \pi_1 \sin k\theta \right \} \]
where the exterior normal of $\partial S_{\text{in}}$ is along $-\mathbf{n}$, to give a sign difference in (3.28). From the orthogonality of trigonometric functions, Eq. (3.28) leads to

\[
\begin{align*}
\left. u_{M-N}\right|_{\partial S_{\text{in}}} = u_{M-N}(\rho, \theta; \mathbf{p}, \mathbf{\bar{p}}) &= a_0 - \left[ R + \ln \left( \frac{\sigma_0}{2} \right) \right] p_0 - \left[ \mathbf{\bar{p}} + \ln \left( \frac{\sigma_1}{2} \right) \right] \mathbf{\bar{p}}_0 \\
&+ \sum_{k=1}^{M} \frac{1}{k} e^{-k\rho} \left\{ \mathbf{p}_k \cos k\theta \cos k\rho + \mathbf{b}_k \sin k\phi \right\} \\
&+ \sum_{k=1}^{M} e^{-k\rho} \left\{ \mathbf{p}_k \cos k\theta \cos k\rho + \mathbf{b}_k \sin k\phi \right\} \\
&+ \sum_{k=1}^{N} \frac{1}{k} e^{-k\rho} \left\{ \mathbf{p}_k \sin k\theta \sin k\rho \cos k\theta \sin k\rho \right\} \\
&+ \sum_{k=1}^{N} e^{-k\rho} \left\{ \mathbf{p}_k \sin k\theta \sin k\rho \cos k\theta \sin k\rho \right\},
\end{align*}
\]

(3.28)

Consider the validity of (3.15) at $\rho \geq R$. For the same exterior field nodes $P(\rho, \theta) = (\mathbf{p}, \mathbf{\bar{p}})$, we have $\mathbf{\bar{p}} \geq R_1$. Hence, there exist the bounds

\[
|e^{-k\rho} \sin k\theta| \leq 1, \quad |e^{-k\rho} \cos k\rho| \leq 1, \quad \rho \leq R,
\]

(3.30)

\[
|e^{-k\rho} \sin k\rho | \leq 1, \quad |e^{-k\rho} \cos k\rho | \leq 1, \quad \mathbf{\bar{p}} \geq R_1.
\]

(3.31)

Eq. (3.15) is valid at $\rho = R$ when $u \in H^2(\partial S_{\text{in}})$ and $u_\nu \in H^1(\partial S_{\text{in}})$ by following the arguments in [25]. We then have the following theorem.

**Theorem 3.1.** Let $u \in H^2(\partial S_{\text{in}})$ and $u_\nu \in H^1(\partial S_{\text{in}})$ be given. Eqs. (3.15) and (3.20) are valid for $(\rho, \theta)$ at $\rho \geq R$ and for $(\mathbf{p}, \mathbf{\bar{p}})$ at $\mathbf{\bar{p}} \geq R_1$, respectively. Moreover, the solution (3.29) is valid for $\rho \leq R$ and $\mathbf{\bar{p}} \geq R_1$.

4. The interior field method

In this section, we propose a new method, based on the interior solution (3.29) directly satisfying the Dirichlet boundary conditions (3.5) and (3.7). Since Eq. (3.29) is derived from (3.27) at interior nodes $Q(\mathbf{p}) \in S$ only, the interior field method (IFM) is called, compared to the null field method (NFMM) from (3.10) at exterior nodes $Q(\mathbf{p}) \in \bar{S}$. The IFM is one of the Trefftz methods in [27]. The IFM was first proposed in [17], and then developed for Neumann problems in circular domains in [20].

Since $\lim_{\rho \to R} f(\rho) = f(\rho_0)$ for any continuous function $f(\rho)$, based on Theorem 3.1, the semi-analytic solutions (3.29) are also valid on the boundary $\partial S$. Then we may seek the unknown coefficients by using (3.29) to satisfy directly the Dirichlet conditions (3.5) and (3.7), as

\[
\begin{align*}
u_{M-N}(\rho, \theta; \mathbf{p}, \mathbf{\bar{p}}) &= a_0 + \sum_{k=1}^{M} \left\{ a_k \cos k\theta + b_k \sin k\theta \right\},
\end{align*}
\]

(4.1)

\[
u_{M-N}(\rho, \theta; \mathbf{p}, \mathbf{\bar{p}}) = a_0 + \sum_{k=1}^{M} \left\{ a_k \cos k\theta + b_k \sin k\theta \right\}. \tag{4.2}
\]

In (3.29), there are $(M+N)+2$ unknown coefficients, and we may choose the same nodes in (3.21) and (3.22) with $e = c = 0$. Denote (4.1) and (4.2) as

\[
\text{IFM}^{\text{M-N}}(\rho, \theta; \mathbf{p}, \mathbf{\bar{p}}) = u_{M-N}(\rho, \theta; \mathbf{p}, \mathbf{\bar{p}}) - a_0 = 0,
\]

(4.3)

\[
\text{IFM}^{\text{M-N}}(\rho, \theta; \mathbf{p}, \mathbf{\bar{p}}) = u_{M-N}(\rho, \theta; \mathbf{p}, \mathbf{\bar{p}}) - a_0 = 0,
\]

(4.4)

We obtain the following collocation equations:

\[
\text{IFM}^{\text{M-N}}(\rho, \theta; \mathbf{p}, \mathbf{\bar{p}}) = 0, \quad j = 0, 1, \ldots, 2M,
\]

(4.5)

\[
\text{IFM}^{\text{M-N}}(\rho, \theta; \mathbf{p}, \mathbf{\bar{p}}) = 0, \quad j = 0, 1, \ldots, 2N.
\]

(4.6)

Where $\Delta \theta = \pi R_1$ and $\Delta \bar{\theta} = \pi R_1$. A similar matrix equation to (3.26) is given, and the unknown coefficients can also be obtained by the Gaussian elimination or some iterative methods. Note that only the solution (3.29) is used in the IFM, while multiple formulas, such as (3.15), (3.20) and (3.29), are used in the NFM. Hence the IFM is simpler than the NFM. Moreover, we have the following lemma.

**Lemma 4.1.** Eq. (4.3) is equivalent to (3.15) as the field nodes $Q \in \partial S_{\text{in}}$, and Eq. (4.4) to (3.20) as $Q \in \partial S_{\text{ex}}$.

**Proof.** We only show the identity between (4.3) and (3.15) as the field nodes located on $\partial S_{\text{ex}}$, since the proof for the identity between (4.4) and (3.20) as $Q \in \partial S_{\text{in}}$ is similar. By substituting (3.29) into (4.3) yields

\[
\begin{align*}
&\left. a_0 - \left[ R + \ln \left( \frac{\sigma_0}{2} \right) \right] p_0 - \left[ \mathbf{\bar{p}} + \ln \left( \frac{\sigma_1}{2} \right) \right] \mathbf{\bar{p}}_0 \\
&+ \sum_{k=1}^{M} \frac{1}{k} e^{-k\rho} \left\{ \mathbf{p}_k \cos k\theta \cos k\rho + \mathbf{b}_k \sin k\phi \right\} \\
&+ \sum_{k=1}^{M} e^{-k\rho} \left\{ \mathbf{p}_k \cos k\theta \cos k\rho + \mathbf{b}_k \sin k\phi \right\} \\
&+ \sum_{k=1}^{N} \frac{1}{k} e^{-k\rho} \left\{ \mathbf{p}_k \sin k\theta \sin k\rho \cos k\theta \sin k\rho \right\} \\
&+ \sum_{k=1}^{N} e^{-k\rho} \left\{ \mathbf{p}_k \sin k\theta \sin k\rho \cos k\theta \sin k\rho \right\},
\end{align*}
\]

(4.7)

On the other hand, when $\rho = R$, Eq. (3.15) leads to

\[
\begin{align*}
\mathbf{\bar{p}}_{\text{ext}}(R, \theta; \mathbf{p}, \mathbf{\bar{p}}) &= - \sum_{k=1}^{M} \frac{1}{k} e^{-k\rho} \left\{ \mathbf{p}_k \sin k\theta \cos k\theta + \mathbf{b}_k \cosh k\theta \right\} \\
&+ \sum_{k=1}^{N} e^{-k\rho} \left\{ \mathbf{p}_k \sin k\theta \cos k\theta \sin k\theta \right\} \\
&+ \sum_{k=1}^{N} \frac{1}{k} e^{-k\rho} \left\{ \mathbf{p}_k \sin k\theta \cos k\theta + \mathbf{b}_k \cosh k\theta \right\}.
\end{align*}
\]
Proof. For the IFM (i.e., the NFM with $\varepsilon = \tau = 0$), there exist the degenerate scale problems if and only if

$$a + b = 2,$$

(4.9)

where $a$ and $b$ are two semiaxes of the exterior elliptic boundary $\partial S_\varepsilon$.

Theorem 4.1. For the IFM (i.e., the NFM with $\varepsilon = \tau = 0$), there exist the degenerate scale problems if and only if

$$R = \frac{\ln (\sigma_0/2)}{\ln (\sigma_1/2)}.$$

(4.10)

where $f_0$ and $\mathcal{P}_0$ are the rest parts of algebraic equations without $p_0$ and $\sigma_0$. Based on the analysis in [21], the sufficient and necessary condition of degenerate scale problems is the zero determinant

$$\begin{vmatrix} \rho + \ln \frac{\sigma_0}{2} & \tau + \ln \frac{\sigma_1}{2} \\ \rho + \ln \frac{\sigma_0}{2} & \tau + \ln \frac{\sigma_1}{2} \end{vmatrix} = 0.$$

(4.11)

For the IFM with $\rho = R$, Eq. (4.11) leads to

$$\begin{vmatrix} R + \ln \frac{\sigma_0}{2} & \tau + \ln \frac{\sigma_1}{2} \\ R + \ln \frac{\sigma_0}{2} & \tau + \ln \frac{\sigma_1}{2} \end{vmatrix} = \left( R + \ln \frac{\sigma_0}{2} \right) (\tau - R) = 0.$$

(4.12)

Since $\tau \geq R_1$ and $\tau \neq R_1$, we have from (4.12)

$$R + \ln \frac{\sigma_0}{2} = 0.$$

(4.13)

When $\theta = 0$ and $\xi$, two semiaxes of the large ellipse $\partial S_\varepsilon$ are given from (3.1)

$$a = \sigma_0 \cosh R, \quad b = \sigma_0 \sinh R.$$

(4.14)

Then we have

$$a + b = \sigma_0 (\cosh R + \sinh R) = \sigma_0 e^R,$$

(4.15)

and then

$$R = \ln \frac{a + b}{\sigma_0}.$$

(4.16)

Eq. (4.13) leads to

$$R + \ln \frac{\sigma_0}{2} = \ln \frac{a + b}{2} = 0,$$

(4.17)

to give (4.9). This completes the proof of Theorem 4.1.

Theorem 4.1 implies that the degenerate scales of the IFM are identical to those of the boundary integral equation (BIE). Theorem 4.1 is consistent with [11,14].

5. The collocation Trefftz method

5.1. The simplest algorithms

For the collocation Trefftz method (CTM), we may choose the particular solutions of polar coordinates in [27]

$$\psi_{\ell k}^{\text{pole}}(r, \phi, \tau, \varphi) = \alpha_0 + \sum_{k = 1}^{M} \left( \frac{R}{R_k} \right)^{\ell} \left( \alpha_k \cos k\phi + \beta_k \sin k\phi \right)$$

$$- e^{-ik\sigma} \psi_{\ell k}^{\text{pole}}(r, \phi, \tau, \varphi) = 0,$$

(5.1)

where $\alpha_k, \beta_k, \tau_k$, and $\sigma_k$ are the coefficients, and $(r, \phi)$ and $(\tau, \varphi)$ are two polar coordinates. By choosing two suitable scale parameters $R$ and $R_1$, the admissible functions (5.1) can be applied to any smooth boundary $\partial S_\varphi \cup \partial S_\tau$. For the domains approximated to an elliptic region, however, it is better to choose the particular solutions in elliptic coordinates. For the elliptic coordinates in (2.1), the Laplace operator can be expressed by (see [29, p. 462])

$$\Delta = \frac{\partial^2 \psi}{\partial r^2} + \frac{\partial^2 \psi}{\partial \theta^2}.$$

(5.2)

Then Laplace’s equation in elliptic coordinates is given by

$$\Delta u = \frac{1}{\sigma^2 (\sin \rho + \sin \theta)} \left( \frac{\partial^2 u}{\partial \rho^2} + \frac{\partial^2 u}{\partial \theta^2} \right) = 0,$$

(5.3)

to give

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{\partial^2 u}{\partial \theta^2} = 0.$$

(5.4)

From the method of separation variables, we obtain the particular solutions

$$1, \rho e^{-\rho} \cos k\theta, \quad e^{-\rho} \sin k\theta.$$

(5.5)

The functions, $\phi_{\ell k}(cos k\theta), \sin k\theta)$ and $e^{-\rho} (\cos k\theta), \sin k\theta)$, are the particular solutions (PS) for the interior and exterior solutions, respectively. We may also choose sinh $k\theta \cos k\theta, \sin k\theta$ and cosh $k\theta \sin k\theta$, instead. For the interior problems, we may also choose (coskphicosk, sinhkphisinhk) instead. Therefore, we have found the following PS for the CTM

$$u_{\ell k + N}^{\text{pole}} = u_{\ell k + N}^{\text{pole}}(\rho, \theta; \tau, \varphi) = \alpha_0 + \tau_0 \varphi$$

$$+ \sum_{k = 1}^{M} e^{-i\kappa} \left( \alpha_k \cosh k \rho \cos k \theta + \beta_k \sinh k \rho \sin k \theta \right)$$

$$+ \sum_{k = 1}^{N} e^{-i\kappa} \left( \tau_k \cosh k \tau_k \cos k \theta + \sigma_k \sinh k \tau_k \sin k \theta \right).$$

(5.6)

Comparing (5.6) with (3.29), there exist the relations of coefficients

$$\alpha_0 = a_0 - \left( R + \ln \left( \frac{\sigma_0}{2} \right) \right) p_0 - \ln \left( \frac{\sigma_1}{2} \right) p_0, \quad \tau_0 = - p_0,$$

(5.7)

$$\alpha_k = a_k + \frac{p_k}{k}, \quad \beta_k = b_k + \frac{q_k}{k}, \quad k = 1, 2, \ldots, M,$$

(5.8)

$$\tau_k = \tau_k + \frac{p_k}{k}, \quad \sigma_k = \sigma_k + \frac{q_k}{k}, \quad k = 1, 2, \ldots, N.$$

(5.9)

The admissible functions (5.6) are the simplest, and they can be applied to any smooth boundary $\partial S_\varphi \cup \partial S_\tau$, see Section 5.3 below.
5.2. No risk of degenerate scales

For Dirichlet problems, the singularity may occur for the boundary integral equation (BIE) under some scaled domain. The reason is that the logarithmic kernel may be trivial when \(\ln r = 0\) at \(r = 1\). This causes troubles in seeking the true solution as well as the numerical solutions, which is called the degenerate scale problems (or degenerate scales). Recently, the study of degenerate scales becomes active, see [7.9–11.13–16.18]. The singularity of the BIE and its numerical algorithms may be different. An advanced study of the NMF in [21] is explored to show more cases leading to the algorithm singularity. To seek numerical solutions, several techniques are proposed in [21], such as the overdetermined system (ODS), the truncated singular value decomposition (TSVD), and scale transformation. To avoid degenerate scales of the NMF, the conservative schemes are also developed in [19]. Unfortunately, there may occur a pseudo-singularity, where the minimal singular value, \(\sigma_{\text{min}} < 1\), of discrete matrices is infinitesimal. The ODS and the TSVD should also be solicited to restore good stability.

There arise questions: What are the rationales to cause the algorithm singularity of the NMF? Can we remove the singularity for Dirichlet problems completely? The degenerate scales are the intrinsic drawback of the BIE, because the FS is the logarithmic kernel as \(\ln r\) in the Green theory of (3.9). Moreover, the FS expansions also involve the logarithmic functions, see Lemma 2.1. Hence, the term causing troubles, as \(a_0 - (R + \ln r)\rho_0 - \psi + (\ln r)\rho_0\), still remains in (3.29) for both IFM and NFM. In fact, the simpler (5.6) are another kind of PS, where two leading functions, 1 and \(\rho^2\), are linearly independent to each other. Consequently, the degenerate scales of the CTM are completely removed.

For circular domains with a circular hole, the interior solutions are given in [25]

\[
\begin{align*}
\nu_{\text{circular}}^{\text{circ}}(r, \phi; \theta, \psi) &= a_0 - R(\ln R)\rho_0 - R_1(\ln R)\rho_0 \\
&+ \frac{R}{2} \sum_{k=1}^{M} \frac{1}{R_k^k} (a_k \cos k\phi + b_k \sin k\phi) \\
&+ \frac{1}{2} \sum_{k=1}^{N} \frac{R_k}{R} (a_k \cos k\phi + b_k \sin k\phi) \\
&+ \frac{R_{\min}}{2} \sum_{k=1}^{M} \frac{1}{R_k^k} (a_k \cos k\phi + b_k \sin k\phi) \\
&+ \frac{1}{2} \sum_{k=1}^{N} \frac{R_k}{R} (a_k \cos k\phi + b_k \sin k\phi) \\
&= a_0 - R(\ln R)\rho_0 - R_1(\ln R)\rho_0, \\
&\text{fully explored in [21]. Comparing (5.10) with (5.1), there exist the relations of leading coefficients}
\end{align*}
\]

(5.10)

The singularity from the leading terms, \(a_0 - R(\ln R)\rho_0 - R_1(\ln R)\rho_0\), is fully explored in [21]. Comparing (5.10) with (5.1), there exist the relations of leading coefficients

\[
\alpha_0 = a_0 - R(\ln R)\rho_0 - R_1(\ln R)\rho_0.
\]

(5.11)

Since two leading basis functions, 1 and \(\ln r\), are linearly independent to each other, the CTM from (5.1) has no risk of degenerate scales, either. Moreover, a better numerical performance by the CTM is given in [17]. For the Helmholtz equation and eigenvalue problems, the CTM in [27, Chapters 9 and 10] can be employed without risk of degenerate scales.

5.3. For non-elliptic domains

Consider the annular domain \(S^\ast\) with the smooth exterior and interior boundaries \(\partial S_{\text{ext}}\) and \(\partial S_{\text{int}}\), respectively. Denote two elliptic coordinates \((\rho, \theta)\) and \((\rho^2, \theta)\) at centers \(O\) and \(\overline{O}\), respectively. The harmonic functions in (5.6) may be modified as

\[
\nu_{\text{annular}} = \nu_{M-N}^{\text{annular}} (\rho, \theta; \rho^2, \theta) = \alpha_0 + \sigma_0 \rho^2
\]

(5.12)

\[\begin{align*}
&\alpha_0 + \sum_{k=0}^{M} \frac{1}{R_k^k} (a_k \cos k\phi + b_k \sin k\phi) \\
&+ \frac{1}{2} \sum_{k=1}^{N} \frac{R_k}{R} (a_k \cos k\phi + b_k \sin k\phi) \\
&+ \frac{R_{\min}}{2} \sum_{k=1}^{M} \frac{1}{R_k^k} (a_k \cos k\phi + b_k \sin k\phi) \\
&+ \frac{1}{2} \sum_{k=1}^{N} \frac{R_k}{R} (a_k \cos k\phi + b_k \sin k\phi) \\
&\in \mathcal{S}^\ast, \\
&\text{where } R_{\min}(\rho > 0) \text{ and } R^\ast_{\min}(\rho > 0) \text{ are two scale parameters. In computation, we may choose}
\end{align*}\]

(5.13)

Note that \(\sigma_{\text{ext}}\) and \(\sigma_{\text{int}}\) are not confined to be elliptic curves. In fact, an annular domain \(S^\ast\) can be embedded into the elliptic domain \(S\) with one elliptic hole, where \(R = R_{\max}\) and \(R_1 = R_{\min}\). When \(R = R_{\max}\) and \(R_1 = R_{\min}\), Eq. (5.12) leads to (5.6). By using the scale parameters, the FS (5.1) of polar coordinates can also be applied to non-circular domains. However, when the annular domain \(S^\ast\) is approximated to elliptic domain \(S\) with an elliptic hole, the particular solutions (5.12) may provide the numerical solutions with high accuracy and good stability. The advantage of (5.12) is significant for the very elongated domains with a very elongated hole.

5.4. Applications to singularity problems

In the CTM [27], the particular solutions, such as (5.1) and (5.2), are chosen to satisfy the boundary conditions. The singularity problems are referred when the solution derivatives are unbounded, which result from the concave corners of solution domains. The interior crack often occurs in applications, see [23,27]. The elliptic coordinates of \(0 < \theta < \Theta\) fail to present the concave corners with \(\Theta < 2\pi\). For an interior crack from \(0 < \theta < 2\pi\), the particular solutions may be found by following [27, Chapter 11]. However, a further study exhibits that such particular solutions fail to present the properties of crack singularity. Hence, the particular solutions in elliptic coordinates are not suitable for singularity problems. We have to solicite the singular solution in polar coordinates. Consider the sectional domain

\[\hat{S} = \{ (r, \phi) | 0 < r \leq R^\ast, 0 < \phi \leq \Theta^\ast \},\]

(5.14)

where \(0 < \Theta^\ast \leq 2\pi\) and (r, \(\phi\)) are polar coordinates. The origin \(O\) is the corner. When \(\Theta = 2\pi\), \(S\) contains an interior crack. For the homogeneous Dirichlet condition on two corner edges

\[u = 0 \text{ at } \phi = 0 \text{ or } \phi = \Theta^\ast\]

(5.15)

the singular solutions (SS) are found in [27, p. 295] as

\[u_{\text{SS}}(r, \phi) = \sum_{k=1}^{M} \beta_k r^\mu \sin k\phi,\]

(5.16)

where \(\mu = \frac{5}{2}\) and \(\beta_k\) are coefficients. The solution derivatives, as \(O(r^\mu - 1)\), from (5.16) are unbounded for \(\mu = \frac{5}{2} < 1\), when \(\Theta > \pi\). The leading coefficient \(\beta_1\) reflects the stress intensity factor, which is one of the most important subjects in fracture mechanics. A complete study of the singular solutions for Laplace’s equations near corners is reported in [27, Chapter 11], where different boundary conditions on the corner edges are considered. The polar coordinates are simpler than the elliptic coordinates. For a concave corner, we can always find a small sectional domain \(S_1\) in (5.14) containing the corner \(O\). Hence, the SS of (5.16) can always be chosen, and all the results for singularity problems in polar coordinates in [23,26,27] can be fully utilized.

In a further study, the elliptic domains (or nearly elliptic domains) with multiple holes and interior cracks are considered, where the boundaries of holes may be circular, elliptic and polygonal. Numerical algorithms of the CTM can be developed.
Let the solution domain $S$ be divided into several subdomains, where the boundaries of the subdomains are not confined to be circular, elliptic or polygonal. The particular solutions, (5.1), (5.12) and (5.16), are chosen in different subdomains. The combined methods are solicited, and the coupling techniques in [23,27] can be employed. Beyond the coupling techniques, the embedding techniques in [22] may be used. For the subdomains with corners, only a few leading singular terms in (5.16) are needed, where small $L$ (say, $L = 2 − 4$) is chosen. The SS of (5.16) are embedded into the smooth solutions, such as (5.1) and (5.12) and the fundamental solutions (FS). High accuracy and good stability of numerical solutions can be achieved, simultaneously. Details appear elsewhere.

5.5. Characteristics and linkages of three boundary methods

There may exist a dilemma for the NFM and the IFM, when the field nodes approach the domain boundary: $Q → ∂S$. Such a dilemma has been clarified by Theorem 3.1 and [20, Section 32]. In this subsection, we will provide more arguments, and exhibit the characteristics and linkages of the NFM, the IFM and the CTM.

If the solution domains are elliptic (or circular), there exist the FS expansions. When the expansions of the FS, the solutions and their derivatives on $∂S$ are substituted into (3.27), the integration can be carried out by manipulation, based on the orthogonality of trigonometric functions. Then we obtain the semi-analytic solutions (3.29), to fall into the “kingdom” of series. Based on the convergence and continuation of series of functions, Eq. (3.29) may hold until the domain boundary $∂S$, given in Theorem 3.1, whose proof is similar to [20, Section 32]. The IFM also steps into the “kingdom” of numerical partial differential equations (PDE). Hence, the IFM owns different characteristics from those of the BIE.

What are the rationales behind these different characteristics? For the direct BIE, there exists the Cauchy singular integral, $\frac{1}{2\pi} \int_{∂S} \frac{u(y) \ln |PQ|}{|PQ|} ds_y = \frac{1}{2\pi} \int_{∂S} u^r(y) \frac{\partial U^s(y)}{\partial r} ds_y$, in (3.9); this is different from the indirect BIE, see [27, Appendix]. Such an integral is converted to series in the NFM and the IFM. Take $∂S_k$ as an example. From (3.27), (3.35) and (4.48), we have from the orthogonality of trigonometric functions

$$\frac{1}{2\pi} \int_{∂S_k} u(y) \frac{\partial |PQ|}{\partial r} ds_y = \frac{1}{2\pi} \int_{∂S_k} u^r(y) \frac{\partial U^s(y)}{\partial r} ds_y$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left\{ a_0 + \sum_{k=1}^{M} \left( a_k \cos k\phi + b_k \sin k\phi \right) \right\} d\phi$$

$$= \left\{ a_0 + \sum_{k=1}^{M} e^{-ik\phi} \left( a_k \cos k\phi \cos \theta + b_k \sin k\phi \sin \theta \right) \right\} \text{ as } \rho \to R.$$ (5.17)

Under the conditions $u \in H^2(∂S_k)$ in Theorem 3.1, it is proved from [20,25] that the decreasing Fourier coefficients have the asymptotes

$$|a_k| = O\left( \frac{1}{k^2} \right), \quad |b_k| = O\left( \frac{1}{k} \right).$$ (5.18)

The last summation of functions in (5.17) is uniformly bounded

$$\lim_{\rho \to R} \left| a_0 + \sum_{k=1}^{M} e^{-ik\phi} \left( a_k \cos k\phi \cos \theta + b_k \sin k\phi \sin \theta \right) \right|$$

$$\leq \left| a_0 \right| + \sum_{k=1}^{M} e^{-ik\phi} \left( |a_k| \cos k\phi \cos \theta | + |b_k| \sin k\phi \sin \theta | \right)$$

$$\leq \left| a_0 \right| + \sum_{k=1}^{M} \left( |a_k| + |b_k| \right) \leq C \sum_{k=1}^{\infty} \left( \frac{1}{k^2} \right) \leq C e^{-\frac{x^2}{y^2}}.$$ (5.19)

where $C$ is a bounded constant independent of $M, \rho$ and $\theta$. Moreover, the functions, $\cos k\rho \cos \theta$ and $\sin k\rho \sin \theta$, are continuous for $\rho \in [0, R]$ and $\theta \in [0, 2\pi]$. Based on the theory of function series, the last summation in (5.17) of functions is continuous until $\rho = R$. Therefore, the semi-analytic solutions (3.29) are continuous until $\rho = R$. The nodes in the IFM can be chosen on the domain boundary. The highly decreasing Fourier coefficients (as (5.18)) can remove the singularity of $\ln |PQ|$ in the Cauchy singular integral. In contrast, such benefits do not exist in the direct BIE. More explorations are given in [20, Section 32], and the error analysis for the IFM in circular domains is provided in [17, Section 4]. The IFM and the NFM “have immigrated” to the “kingdoms” of series and PDE, so that not all rules of the direct BIE are valid.

The IFM is the special NFM when nodes $Q \in \partial S$, based on Lemma 4.1. For the CTM, the coefficients $a_k, b_k, \pi_k$ and $\beta_k$ in (5.6) are not the Fourier coefficients of the solutions. The CTM can be obtained from the IFM under the coefficient relations (5.7)–(5.9). Therefore, the CTM may be regarded as an advanced version.
of the IFM, to have no risk of degenerate scales (see Section 5.2),
and wide applications for non-elliptic domains (see Section 5.3).
Also the CTM can be applied to singularity problems (see Section 5.4).
Here the intrinsic linkages of these three boundary methods are
found.
To easily catch the essentials of the numerical methods
proposed, let us give a vivid simile with butterfly and pupa. A
butterfly sloughes off its pupa. The pupa is confined not to touch a
boundary, and suffers from some illness. The butterfly can fly, and
may reach the boundary. The NFM (as a butterfly) originated from
the direct BIE (as a pupa) under the elliptic (or circular) domains,
but can cross its boundary, to become the IFM and enter the PDE.
The CTM (as an adult butterfly, i.e., imago) is an advanced version
of the IFM. The adult butterfly becomes healthy and beautiful,
because the inherent illness (e.g., degenerate scales) disappears,
and because she gains robustness (e.g., for non-elliptic domains).
The boundary value problems of Laplace’s equation were first
converted to the BIE by George Green in 1828 (see [27, Appendix,
p. 334]), and vice versa in this paper. This illustrates a life cycle
of butterfly and pupa; amazing!

6. Numerical experiments

6.1. The IFM

Consider a large elliptic domain with a small elliptic hole, as
shown in Fig. 3. The large ellipse is defined by \(a = 2.5, b = 2,\)
\(\sigma_0 = \sqrt{a^2 - b^2} = 1.5\) and
\[x = \sigma_0 \cosh \theta R \cos \psi, \quad y = \sigma_0 \sinh \theta \sin \psi.\]  
(6.1)
From \(\tanh \theta = \frac{b}{a} = \frac{2}{5},\) we have \(R = 1.0986122887.\) The small
ellipse is defined by another elliptic coordinates with origin
\((0, 0, 0).\)
\[x = \sigma_1 \cosh \theta R \cos \psi - 1, \quad y = \sigma_1 \sinh \theta R \sin \psi,\]  
(6.2)
where \(\sigma_1 = 1, \theta = 45^\circ,\) and \(\psi = \sqrt{\sigma_1^2 - 2} = \frac{\sqrt{2}}{5}.\) For simplicity in com-
putation, the long axis of the small ellipse is chosen along semiaxis
\(x\) with \(\theta = 0.\) From \(\tanh \theta R_1 = \frac{5}{4},\) we have \(R_1 = 0.54930614433.\)
The Dirichlet condition is given by
\[u = 0\]  
\(0\) on \(\partial S_R, \quad u = \sigma_0\) on \(\partial S_R_1.\)
(6.3)
Such a problem is called Model Problem in this paper.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Errors and condition numbers for Model Problem by the IFM with (M = 18) and various (N.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(M )</td>
<td>(N )</td>
</tr>
<tr>
<td>18</td>
<td>4</td>
</tr>
<tr>
<td>18</td>
<td>5</td>
</tr>
<tr>
<td>18</td>
<td>6</td>
</tr>
<tr>
<td>18</td>
<td>7</td>
</tr>
<tr>
<td>18</td>
<td>8</td>
</tr>
<tr>
<td>18</td>
<td>9</td>
</tr>
<tr>
<td>18</td>
<td>10</td>
</tr>
<tr>
<td>18</td>
<td>12</td>
</tr>
</tbody>
</table>

From the symmetry of Model Problem, the interior solutions
(3.29) lead to
\[u_{int} = -R + \ln \left( \frac{\sigma_R^2}{\omega} \right) p_{R} - \frac{\theta}{\kappa} \left[ \arctan \left( \frac{\sigma_R^2}{\omega} \right) \right] p_{0} \]
\[+ \sum_{k=1}^{M} \frac{A_{k}}{k} e^{ikR} \cos k\theta \cos k\phi + \sum_{k=1}^{N} \frac{B_{k}}{k} e^{ik\theta} \sin kR \cos k\theta.\]  
(6.4)
When Eq. (6.4) satisfies (6.3), we have the following two boundary
equations:
\[\ell_{out}(R, \theta, \phi, \partial) = \left[ R + \ln \left( \frac{\sigma_0^2}{\omega} \right) \right] p_{0} - \frac{\theta}{\kappa} \left[ \arctan \left( \frac{\sigma_0^2}{\omega} \right) \right] p_{0} \]
\[- \sum_{k=1}^{M} \frac{A_{k}}{k} e^{ikR} \cos k\theta \cos k\phi - \sum_{k=1}^{N} \frac{B_{k}}{k} e^{ik\theta} \sin kR \cos k\theta = 0.\]  
(6.5)
\[ \varepsilon = u - u^{\text{opt}}_{\Delta N} \]

Table 6

Errors and condition numbers for Model Problem by the CTM with \( M = 3N \), where \( \varepsilon = u - u^{\text{opt}}_{\Delta N} \).

<table>
<thead>
<tr>
<th>( M )</th>
<th>( N )</th>
<th>( |\varepsilon|_{0,05} )</th>
<th>( |\varepsilon|_{\infty,05} )</th>
<th>Cond(A)</th>
<th>Cond_{\text{eff}}(A)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>2</td>
<td>1.02 (−2)</td>
<td>9.16 (−1)</td>
<td>15.63</td>
<td>4.11</td>
</tr>
<tr>
<td>12</td>
<td>4</td>
<td>4.34 (−4)</td>
<td>3.64 (−4)</td>
<td>15.62</td>
<td>4.15</td>
</tr>
<tr>
<td>18</td>
<td>6</td>
<td>2.29 (−5)</td>
<td>2.01 (−5)</td>
<td>15.58</td>
<td>4.15</td>
</tr>
<tr>
<td>24</td>
<td>8</td>
<td>1.38 (−6)</td>
<td>1.29 (−6)</td>
<td>15.56</td>
<td>4.16</td>
</tr>
<tr>
<td>30</td>
<td>10</td>
<td>9.02 (−8)</td>
<td>8.87 (−8)</td>
<td>15.55</td>
<td>4.16</td>
</tr>
</tbody>
</table>

Table 7

The coefficients \( a_k \) and \( \pi_k \) of the CTM in Table 6 at \( (M : N) = (30 : 10) \).

<table>
<thead>
<tr>
<th>( k )</th>
<th>( a_k )</th>
<th>( \pi_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1.13083643441242</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>4.0548468285728 (−1)</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>5.424019062596221 (−3)</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>6.8535436533494 (−5)</td>
<td>9</td>
</tr>
<tr>
<td>5</td>
<td>7.227708798333 (−4)</td>
<td>10</td>
</tr>
</tbody>
</table>

\[ \dot{L}_{\text{int}}(\rho, \theta R_k, \bar{t}) = -a_0 + \left[R + \ln\left(\frac{\sigma}{2}\right)\right]p_0 + \left[R + \ln\left(\frac{\sigma}{2}\right)\right]p_0 - \sum_{k=1}^{M} p_k e^{-kr} \cosh k\rho \cos k\theta - \sum_{k=1}^{N} \frac{p_k e^{-kr} \cosh k\rho \cos k\theta}{k}. \]  

(6.6)

Based on Lemma 4.1, Eqs. (6.5) and (6.6) can also be obtained from (3.15) as \( \rho = 1 \) and from (3.20) as \( \rho = 1 \), respectively. In (6.5) and (6.6), there are \( (M + N + 2) \) unknown coefficients of \( p_k \) and \( \pi_k \). We choose the following collocation equations:

\[ w_j \dot{L}_{\text{ext}}(R_j, \Delta \theta; \bar{t}_j, \bar{t}_{j}) = 0, \quad j = 0, 1, \ldots, M, \]  

(6.7)

\[ w_j \dot{L}_{\text{int}}(\rho_j, \theta; R_j, \Delta \bar{t}) = 0, \quad j = 0, 1, \ldots, N, \]  

(6.8)

where \( \Delta \theta = \frac{2\pi}{M+1} \) and \( \Delta \bar{t} = \frac{2\pi}{N+1} \). In (6.7) and (6.8), the weights \( w_0 = 1 \) and \( w_j = \sqrt{2}/(N+1) \) are chosen for better stability, based on the analysis in [20]. The effective condition number and the effective condition number in [26] are defined by

\[ \text{Cond} = \frac{\sigma_{\text{max}}}{\sigma_{\text{min}}}, \quad \text{Cond}_{\text{eff}} = \frac{\sigma_{\text{max}}}{\sigma_{\text{eff}}} \]  

(6.9)

where \( \| \cdot \|_2 \) is 2-norm, and \( \sigma_{\text{max}} \) and \( \sigma_{\text{min}} \) are the maximal and the minimal singular values of matrix \( A \) in (3.26), respectively. Let \( \varepsilon = u - u^{\text{opt}}_{\Delta N} \), and define the error

\[ \|\varepsilon\|_{0,05} = \left( \|\varepsilon\|_{0,05}^2 + \|\varepsilon\|_{\infty,05}^2 \right)^{1/2}. \]  

(6.10)

By the IFM (i.e., the NFM at \( \varepsilon = \tau = 0 \)), the errors and the condition numbers are listed in Tables 1 and 2, and the coefficients in Table 3. In Table 1, the condition numbers are found as \( \sum = 3 \). From Table 2, we may find the following asymptotes:

\[ \|\varepsilon\|_{0,05} = O(0.63M), \quad \|\varepsilon\|_{\infty,05} = O(0.64M). \]  

(6.11)

\[ \text{Cond}(A) = O(M), \quad \text{Cond}_{\text{eff}}(A) = O(M). \]  

(6.12)

Eqs. (6.11) and (6.12) indicate the highly accurate solutions and very good stability.

6.2. The NFM with \( \varepsilon > 0 \) and \( \tau > 0 \)

Next, choose the NFM with nodes as shown in Fig. 4, where \( \varepsilon = (0.2, 0.05) \) and \( \tau = (1.2, 0.05) \), and list the results in Tables 4 and 5. Compared Table 4 with Table 2, the condition numbers are significantly larger, yet the errors are slightly smaller. Moreover, the condition numbers in Table 5 grow exponentially. When \( (M : N) = (30, 10) \), a huge Cond = 1.34(15) causes the large error, as \( \|\varepsilon\|_{0,05} = 0.36 \). Hence, the IFM (i.e., the NFM with \( \varepsilon = \tau = 0 \)) is better than the NFM with \( \varepsilon = \tau > 0 \). This conclusion coincides with the results in [25].

6.3. The CTM

In the CTM, from (5.6) we choose the following particular solutions:

\[ u^{\text{opt}}_{\Delta N} = a_0 + \pi_0 \bar{t} + \sum_{k=1}^{M} e^{-kr} a_k \cosh k\rho \cos k\theta \]  

(6.13)

The collocation boundary equations are also obtained directly from (6.3). This is exactly the CTM in [27]. The results are listed in Tables 6 and 7. From Table 6, the errors are the same as (6.11), but the condition numbers are constant

\[ \text{Cond}(A) = O(1), \quad \text{Cond}_{\text{eff}}(A) = O(1), \]  

(6.14)

to display the excellent stability. Note that the coefficients in Tables 6 and 7 satisfy the relations (5.7)–(5.9).

7. Concluding remarks

Let us give a few remarks to address the novelties of this paper:

1. The infinite series expansions of the FS are important not only to the algorithms of the NFM, but also to the error analysis for the method of fundamental solutions (MFS), see [24]. In Section 2, the elliptic coordinates are introduced, and the FS expansions are derived in detail. Moreover, the explicit formulas of transformation between two elliptic coordinates are provided in Section 2.2.

2. For elliptic domains, the explicit algebraic equations of the NFM are derived. The explicit equations are significant to both computation and analysis, see [17,25], Theorem 3.1 implies that the field nodes may approach to the elliptic boundaries (i.e., \( \varepsilon = \tau = 0 \)), as shown in circular domains in [25].

3. The new interior field method (IFM) of [17,20] in circular domains is developed for elliptic domains, and the explicit equations are derived in Section 4. In the IFM, only one interior field equation \( u_{\Delta N} \) in (3.29) is used. Lemma 4.1 displays that
the IFM is equivalent to the special case of NFM at $\epsilon = \tau = 0$. Hence, we may replace the NFM by the simple IFM for a wide variety of applications.

4. For the Dirichlet problems by the IFM and the NFM, there does exist the algorithm singularity, as shown in [21]. The degenerate scales occur if the semiaxes of the exterior ellipse satisfy $a + b = 2$, from Theorem 4.1. In this case, the truncated singular value decomposition (TSVD) and the overdetermined systems are recommended, also see [21].

5. For elliptic domains with elliptic holes, the collocation Trefftz methods (CTM) in [17,27] can be chosen, where the particular solutions (PS) are used to satisfy the Dirichlet boundary conditions. The remarkable advantages of the CTM are no risk of degenerate scales given in Section 5.2, and validity for non-elliptic domains given in Section 5.3. In Section 5.4, a brief outline is provided how to deal with singularity problems.

6. In Section 5.5, more arguments are provided for the interior solutions as nodes $Q \rightarrow \alpha \delta$. The characteristics and the linkages of the NFM, the IFM and the CTM are explored, and a metaphor is given, to exhibit the essentials of three methods in this paper.

7. Although there exist many reports of the NFM for circular domains, only a few for elliptic domains [12,28,29]. For Laplace's equation in elliptic domains, a comparative study of algorithms, errors, stability and numerical results is explored in this paper for three boundary methods: the NFM, the IFM and the CTM.

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